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MATHEMATIK

Edge-Dominating Trails in AT-free Graphs

by

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No. 615/1998

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December 21, 1998

Abstract

A triple of independent vertices of a graph G is called an *asteroidal triple* (AT) if between any two of them there exists a path in G that does not intersect the neighborhood of the third.

In this paper we consider different classes of graphs that are related to AT-free graphs. We start by examining AT-free line graphs, give a characterization of them, and apply this for showing that all connected AT-free line graphs are traceable. In the second part we consider line graphs of AT-free graphs. Here we prove that every AT-free graph contains an edge-dominating trail, and that, consequently, every line graph of an AT-free graph is traceable. Moreover, we give an algorithm to find such an edge-dominating trail. In the third part of the paper we consider claw-free AT-free graphs and show a couple of Hamiltonian properties for them, using the RYJÁČEK closure. In the last section we give a characterization of all AT-free graphs with maximum degree at most 3.

1 Introduction

The purpose of this work is to gain deeper insight into the structure of AT-free graphs. They are known to have nice vertex-domination properties [5, 7], and we shall extend these investigations to edge-domination, with particular emphasis on edge-dominating trails.

Since an edge-dominating trail of some graph indicates a hamiltonian cycle in its line graph, it is worthwhile to have a look at line graphs of AT-free graphs. Whereas it is not true that the line graph of some AT-free graph is AT-free again, the converse holds: The AT-free line graphs form a subclass of the line graphs of AT-free graphs, and it is even possible to give a full characterization of them.

[*]Supported by the graduate school 'Algorithmische Diskrete Mathematik', grant GRK 219/2-97 of the Deutsche Forschungsgemeinschaft.

[†]Parts of this work have been written during the second author's visit at the University of Twente, Enschede, The Netherlands.

Another application of the results on edge-domination is the characterization of all 2-connected AT-free graphs with maximum degree at most 3. One could hope for such a characterization, since a sufficiently highly connected graph with small maximum degree and enough vertices fails to contain large induced cycles only under very particular (local) conditions. We use this to characterize all 3-regular AT-free graphs.

For terms not defined here we refer to [1] or [8]. In this paper we consider finite undirected graphs $G = (V, E)$ without loops or multiple edges. The cardinality of the vertex set V is denoted by n and will be refered to by $|G|$, and the cardinality of the edge set E is denoted by m.

To indicate that two graphs G_1, G_2 are isomorphic, we use the notation $G_1 \cong G_2$. For every vertex $v \in V$ we denote by $\mathrm{N}(v)$ the set of all neighbors of v, $\mathrm{N}(v) = \{u \in V : (u, v) \in E\}$. The *closed neighborhood* of v is defined by $\mathrm{N}[v] = \mathrm{N}(v) \cup \{v\}$. The *complement* $\overline{G} = (\overline{V}, \overline{E})$ of G is defined to be the graph with vertex set $\overline{V} = V$, and with $(u, v) \in \overline{E}$ if and only if $(u, v) \notin E$.

An *articulation vertex* $x \in V$ is a vertex that separates some component of G. A *block* of a graph is a maximally induced subgraph without articulation vertices, and we call it *non-trivial* if it is non-isomorphic to a K_2 or a K_1. A component of G is called a *non-trivial component* if it contains at least two vertices.

A set $U \subset V$ is called an *independent set of vertices*, if $(u, v) \notin E$ for all $u, v \in U$, and a set of edges $D \subset E$ is called an *independent set of edges*, if every pair of edges of D does not have an end vertex in common.

A (not necessarily induced) subgraph $P = (V_P, E_P)$ of G is called a *path* of length k, if $V_P = \{x_0, x_1, \ldots, x_k\}$, $E_P = \{(x_0, x_1), (x_1, x_2), \ldots, (x_{k-1}, x_k)\}$. Similarly, a subgraph $C = (V_C, E_C)$ of G is called a *cycle* of length k if $V_C = \{x_0, x_1, \ldots, x_{k-1}\}$, $E_C = \{(x_0, x_1), (x_1, x_2), \ldots, (x_{k-2}, x_{k-1}), (x_{k-1}, x_0)\}$. A set of edges $T = \{e_1, e_2, \ldots, e_k\}$ of G is called a *trail* of length k, if $e_i \neq e_j$ for all $i \neq j$ and for each i with $1 < i < k$ one of the end vertices of e_i equals one of the end vertices of e_{i-1}, whereas the other end vertex of e_i is equal to one of the end vertices of e_{i+1}; T is called a *circuit* of length k if this property holds also for the edge e_k, where e_{k+1} is set to e_1. We refer to the *length* k of a path, cycle, trail, circuit K, respectively, by $\mathrm{length}(K)$.

A subset D of V or a subgraph $H = (D, X)$ of $G = (V, E)$ is said to be a *(vertex-) dominating set* or a *dominating subgraph*, respectively, if every vertex in $V - D$ is adjacent to some vertex in D. A set X of edges of E is said to be a *(vertex-) dominating set* if the set of the end vertices of its members is a dominating set. If D is a path, cycle, trail, circuit, respectively, of G, it is called a *dominating path, cycle, trail, circuit*, respectively. A subset D of V or a subgraph $H = (D, X)$ of $G = (V, E)$ is said to be an *edge-dominating set* or an *edge-dominating subgraph*, respectively, if every edge in E is incident to some vertex in D. *Edge-dominating paths, cycles, trails, circuits* are defined correspondingly.

A *hamiltonian cycle* or *hamiltonian path* of G is a cycle or path, respectively, containing all vertices of G. G is called *hamiltonian* if it has a hamiltonian cycle, and it is called *traceable* if it has a hamiltonian path. If, moreover, between any two vertices of G there exists a hamiltonian path, then G is called *hamiltonian connected*. G is called k*-pancyclic* if it contains cycles of every length $\ell \in \{k, k+1, \dots, |V(G)|\}$.

The *line graph* of a graph G, denoted by $\mathrm{L}(G)$, is the graph whose vertices are the edges of G, where (e, f) is an edge of $\mathrm{L}(G)$ whenever $e = (u, v)$ and $f = (v, w)$ are edges of G.

The four-vertex star $K_{1,3}$ is called the *claw*, and a graph G is called *claw-free* if it does not contain the claw as an induced subgraph.

An *asteroidal triple* or, briefly, an AT of G is a set of three independent vertices such that each pair of vertices is joined by a path not containing vertices of the neighborhood of the third vertex. Consequently, a graph G is called *asteroidal triple-free* or AT*-free* if there is no asteroidal triple in G.

For graphs G, H let $H * G$ arise by taking two vertex disjoint copies of G, H, respectively, and then add all edges between pairs of vertices in $V(G) \times V(H)$.

2 The AT-free line graphs

This section is concerned with the problem of determining all AT-free line graphs. As it turns out, the property of $\mathrm{L}(G)$ not to contain an AT corresponds to a simpler property of G, at least in case that $\mathrm{L}(G)$ is 2-connected:

Lemma 2.1 *Let G be a 2-connected graph. Then $\mathrm{L}(G)$ contains an AT if and only if G contains 3 independent edges.*

Proof. If $\mathrm{L}(G)$ contains an AT then it contains 3 independent vertices, which correspond to 3 independent edges in G.

Conversely, let e, f, g be 3 independent edges in G. If any two of them are in the same component of the graph obtained from G by removing the end vertices of the third, then e, f, g form an AT in $\mathrm{L}(G)$. Otherwise, without loss of generality, $V(e)$ separates f from g. By Menger's Theorem, there are two disjoint paths between $V(e)$ and $V(f)$, and two disjoint paths between $V(e)$ and $V(g)$. Since $V(f)$ and $V(g)$ are in distinct components of $G - V(e)$, the edges of all four paths together with f and g form a cycle in G of length at least 6. This cycle forms a *chordless* cycle in $\mathrm{L}(G)$ — so $\mathrm{L}(G)$ contains an AT. $\square$

Lemma 2.2 *Let G be a 2-connected graph. Then G contains no set of 3 pairwise independent edges if and only if $|G| \leq 5$, or $G \cong K_2 * \overline{K_n}$, or $G \cong \overline{K_2} * \overline{K_n}$ ($n \geq 4$).*

Proof. It is easy to see that none of the exceptional graphs contains a set of 3 independent edges.

Suppose that G contains no set of 3 independent edges.

If G contains a cycle C of length 5 then $V(C) = V(G)$, for otherwise there would be an edge (x, y) with $x \in V(C)$ and $y \in V(G) - V(C)$ and two independent edges in $C - \{x\}$, which form a set of 3 independent edges.

Since any cycle on more than 5 vertices contains 3 independent edges, we may assume that each cycle of G has length 3 or 4.

If G contains no cycle of length 4, then G must be a triangle (so a graph $K_2 * K_1$). Thus we may assume that there is a cycle C of length 4. If $|G| = 4$ then either $G \cong K_2 * \overline{K_2}$, or $G \cong \overline{K_2} * \overline{K_2}$, or $G \cong K_4$.

Hence, we may assume that there is a vertex $z \in V(G) - V(C)$. Since G contains no cycle of length 5, z can not have two neighbors in C which are adjacent in C. Since $G - C$ contains no edges, x has degree 2 and is adjacent to two vertices $u, v \in C$ which are non-adjacent in C. Let u', v' be the vertices in $C - \{u, v\}$. Since G contains no cycle of length 5, there is no edge between v' and u'. Since G contains no cycle of length 6, there is no vertex $z' \in G - C$ adjacent to u', v'. Consequence: If there is an edge between u and v then $G \cong K_2 * \overline{K_n}$, $n \geq 4$, otherwise $G \cong \overline{K_2} * \overline{K_n}$. □

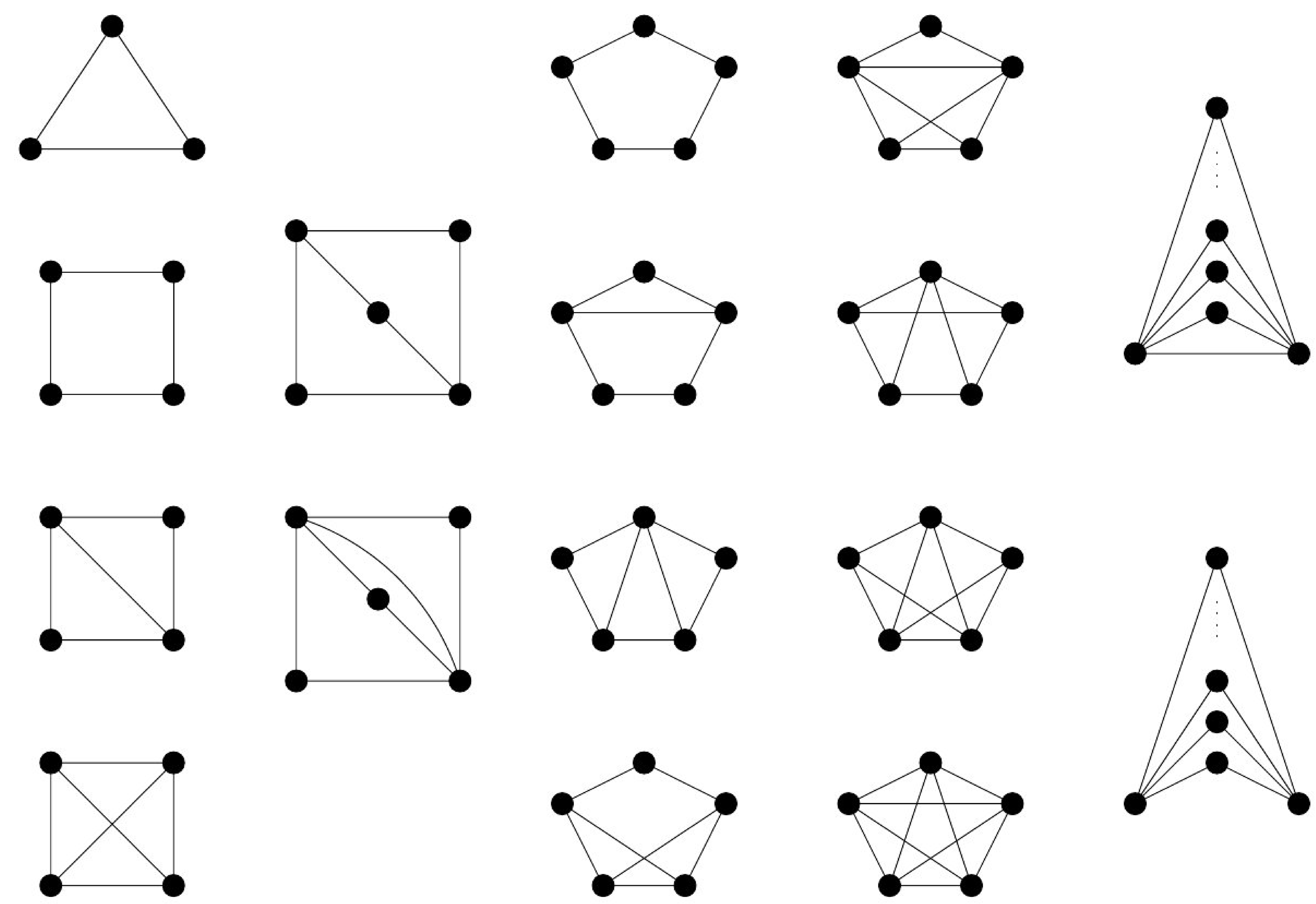

Figure 1: All 2-connected graphs not containing 3 pairwise independent edges.

For the case of 2-connected graphs, the two preceding lemmata provide already a full characterization of all asteroidal triple-free line graphs:

Theorem 2.3 *Let G be a 2-connected graph. Then $\mathrm{L}(G)$ is AT-free if and only if $|G| \leq 5$, or $G \cong K_2 * \overline{K_n}$, or $G \cong \overline{K_2} * \overline{K_n}$ $(n \geq 4)$.*

Proof. Straightforward from Lemma 2.1 and Lemma 2.2. □

Characterizing not only the 2-connected but *all* asteroidal triple-free line graphs takes a bit more effort. As one could expect, Lemma 2.1 and Lemma 2.2 are quite useful for this purpose.

Theorem 2.4 *Let G be a connected graph. Then $\mathrm{L}(G)$ is AT-free if and only if $|G| \leq 5$ or G can be obtained from a path P of length ≥ 0 by replacing each edge of P by a graph $K_2 * \overline{K_n}$, $n \geq 0$, or a graph $\overline{K_2} * \overline{K_n}$, $n \geq 2$, such that the end vertices of the former edge correspond to the end vertices of some K_2 or $\overline{K_2}$ on the left hand side of the product, and then link an arbitrary number of new pendant vertices each to precisely one vertex of the former path P. Alternatively, one or two of the end edges of P can be replaced by a K_4, but in this case no pendant vertex can be linked to the corresponding end-vertex (or to the corresponding end-vertices) of P.*

Proof. Let G be a graph as in the assertion and suppose that $e, f, g \in E(G)$ form an AT in $\mathrm{L}(G)$. By Lemma 2.2, they are not all contained in the same block of G, and each of them dominates some articulation vertex unless they are contained in a K_4 one of the end edges has been replaced with. Without loss of generality, f dominates thereby some articulation vertex x such that e, g are in distinct components of $G - x$. It follows that every e, g-path in $\mathrm{L}(G)$ intersects some vertex of $\mathrm{N}_{\mathrm{L}(G)}(f)$, and thus e, f, g do *not* form an AT — a contradiction.

Now let us suppose that $\mathrm{L}(G)$ is AT-free. We shall see that G is of the form proposed in the assertion.

If G is 2-connected graph then this follows from Lemma 2.3. So we may suppose that G contains at least one articulation vertex.

Claim 1. For $x \in V(G)$, $G - x$ has at most two non-trivial components, for if there were three such components then we take one edge from each of them; these three edges form an AT in $\mathrm{L}(G)$.

Claim 2. Each block H of G contains at most 2 articulation vertices of G, for otherwise there were three independent edges with one end vertex in $V(H)$ and one in $V(G) - V(H)$. These edges form an AT in $\mathrm{L}(G)$ as well.

By Claim 1 and Claim 2 it suffices to show

Claim 3. each block H is of the form $K_2 * \overline{K_n}$, $n \geq 0$, or of the form $\overline{K_2} * \overline{K_n}$, $n \geq 2$, or of the form K_4 provided that it is an end block, and

Claim 4. only the vertices of a K_2 or a $\overline{K_2}$ on the left or right hand side of the product may be articulation vertices.

To prove Claim 3, assume for a while that H is not of the form mentioned in there. By Lemma 2.2 we know that $|H| \leq 5$, and thus either $|G| = 5$ or $H \cong K_4$ and H contains two articulation vertices of G.

The latter case is impossible, since otherwise there would be independent edges $e, f \in E(G) - E(H)$ incident with two articulation vertices of G in $V(H)$ and some further edge g in $H \cong K_4$ not incident with e or f. It is easy to see that e, f, g form an AT in $\mathrm{L}(G)$.

So $|H| = 5$, and, by assumption and Lemma 2.2, H must contain a hamiltonian cycle C.

Since G is not 2-connected, C contains an articulation vertex of G, say c. Consider an edge leading from c to $G - V(C)$, and two independent edges of $C - c$. It is then easy to see that the these three edges form an AT in $\mathrm{L}(G)$. This contradiction proves Claim 3.

To prove Claim 4, suppose that there is an articulation vertex c of G contained in H which corresponds to a vertex of a $\overline{K_n}$, $n \geq 3$ as a right hand side factor in the product H is equal to. Then again we may consider an edge leading from c to $V(G) - V(H)$ and two independent edges of $H - c$. These three edges form an AT in $\mathrm{L}(G)$. □

Note that the graphs described in the preceding theorem all possess an edge-dominating path, and, in the case that they contain no non-trivial bridges (i.e. bridges leaving two non-trivial components), even an edge dominating cycle. Using the following well known characterization of traceable and hamiltonian line graphs, we can readily take advantage of the above result for characterizing all traceable, resp. hamiltonian, asteroidal triple-free line graphs.

Lemma 2.5 ([10]) *Let G be a graph without isolated vertices.*

(i) $\mathrm{L}(G)$ is traceable if and only if G contains a dominating trail, and

(ii) $\mathrm{L}(G)$ is hamiltonian if and only if $G \cong K_{1,n}$, for some $n \geq 3$, or G contains a dominating circuit.

Together with Theorem 2.4, we obtain

Theorem 2.6 *Let G be an AT-free line graph.*

(i) G is traceable if and only if G is connected and

(ii) G is hamiltonian if and only if it is 2-connected.

Theorem 2.6 immediately follows from a result of Shepherd [14], as we shall see later on.

Theorem 2.7 *Let H be a 2-connected AT-free line graph and $x \neq y$ in $V(H)$. Then there exists a hamiltonian x,y-path in H if and only if $H - \{x, y\}$ is connected.*

Proof. Let G be a graph such that $\mathrm{L}(G) = H$ is 2-connected. We may assume that $\mathrm{L}(G)$ is non-complete. In particular, G is not a star and therefore contains at least one non-trivial block.

G arises from a path $x_1, \ldots, x_n$, $n \geq 1$, as described in Theorem 2.4. We may take a path from which G arises as there such that n is as small as possible. Let H_i be the block containing x_i, x_{i+1} for $i \in \{1, \ldots, n-1\}$.

We may assume that the blocks H_1, H_{n-1} are non-trivial, for otherwise we could produce G also from the path $x_2, \ldots, x_n$, or from the path $x_1, \ldots, x_{n-1}$, respectively. Since G contains no non-trivial bridges, it follows $|H_i| \geq 3$ for all $i \in \{1, \ldots, n-1\}$.

Let e, f be edges of G such that $\mathrm{L}(G - \{e, f\})$ is connected, i.e. $G - \{e, f\}$ has at most one non-trivial component.

We find two edge disjoint x_i, x_{i+1}-paths $P_{i,1}, P_{i,2}$ in H_i such that both $P_{i,1}$ and $P_{i,2}$ dominate the edge set. Let $P_j = \bigcup_{i=1}^{n-1} P_{i,j}$ for $j \in \{1, 2\}$.

We try to choose $P_{i,1}$, $P_{i,2}$ in such a way that the case $e \in P_{i,1}$, $f \in P_{i,2}$ (or vice versa) does not occur.

This is possible unless $H_i \cong K_2 * K_1$ or $H_i \cong \overline{K_2} * \overline{K_2}$ or $H_i \cong \overline{K_2} * K_2$, where x_i, x_{i+1} correspond to the vertices of the left hand side factors of the products. By the fact that $G - \{e, f\}$ has at most one non-trivial component it turns out that e, f have to be incident with a common vertex of degree 2 in H and in G, and thus incident with x_1 or x_n. In these cases, the EULER subgraph $P_1 \cup P_2$ contains an edge-dominating trail with end edges e, f, which gives raise for a hamiltonian x, y-path in $\mathrm{L}(G)$.

Therefore, we may assume that $e \notin P_2$ and that $f \notin P_2$. Without loss of generality, let the length of the $x_1, V(e)$-subpath P_{1a} of P_1 be at most the length of the $x_1, V(f)$-subpath of P_1. Then P_{1a} and the $V(f), x_n$-subpath P_{1b} of P_1 are edge disjoint and contain neither e nor f. Therefore, the edges of $e, P_{1a}, P_2, P_{1b}, f$ form an edge-dominating trail with end edges e, f, from which one can construct a hamiltonian x, y-path in $\mathrm{L}(G)$. □

From the preceding theorem, Theorem 2.6 follows, too: Suppose that G is an AT-free 2-connected graph on at least 4 vertices, not necessarily a line graph. As it is well-known, such a graph contains a *contractible edge* e, i.e. an edge whose contraction yields again a 2-connected graph. Therefore, $G - V(e)$ is connected, and so there exists a hamiltonian path between the end vertices of e. Another Corollary of Theorem 2.7 is the following.

Theorem 2.8 *Let G be an AT-free line graph. Then G is hamiltonian connected if and only if it is 3-connected or a triangle.*

3 Edge-dominating trails and circuits in AT-free graphs

When studying asteroidal triple-free graph in relation to line graphs, it is quite a natural question to ask whether the property of being asteroidal triple-free is maintained by the process of forming the line graph of a given graph G and, vice versa, whether the graph G is asteroidal triple-free if the corresponding line graph is.

The first question is easily answered by looking at the complete graph on six vertices. Obviously, it is asteroidal triple-free, since it does not even have an independent triple of vertices. Its line graph, on the other hand, has many asteroidal triples, since $\mathrm{L}(G)$ contains 6! induced cycles on six vertices. (Any independent triple of such a cycle forms an asteroidal triple.) The second question is answered by the following theorem.

Theorem 3.1 *Let $\mathrm{L}(G)$ be the line graph of some graph G. If $\mathrm{L}(G)$ is AT-free then G is AT-free.*

PROOF. Suppose G contains an AT x, y, z. Let P_1 be the path between x and y, avoiding the neighborhood of z, P_2 the path between y and z, avoiding the neighborhood of x, and P_3 the path between z and x, avoiding the neighborhood of y. Let e_i be the first edge on path P_i (e_1 is incident to x, e_2 to y and e_3 to z). P_1 avoids the neighborhood of z, hence none of the edges of P_1 is incident to e_3. Likewise e_2 is not incident to any edge of P_2 and, consequently, not incident to e_3 either. The edges of P_1 together with e_2 form an e_1e_2 path in $\mathrm{L}(G)$ and by the above remarks, this path avoids the neighborhood of e_3 in $\mathrm{L}(G)$. Analogously one can prove the existence of the corresponding e_2e_3 and e_3e_1 paths. Hence $\mathrm{L}(G)$ has an AT. □

When we now start to analyze the properties of line graphs of asteroidal triple-free graphs, the preceding observation assures that we are considering a proper superclass of asteroidal triple-free line graphs. The question arises, whether this larger class still has such a clear structure that enables us to find hamiltonian paths and cycles as easily as in the previous case.

By Theorem 2.4, each connected graph that has an asteroidal triple-free line graph does have an edge-dominating trail as well. A closer look at this characterization reveals that it even has an induced edge-dominating *path.* On the other hand, for asteroidal triple-free graph in general it was shown by Corneil et al. [5, 7] that each such graph G does have a dominating path, even stronger, it contains a *dominating pair*, i.e. there is a pair of vertices x, y in G such that *every* path, connecting x and y is a dominating path of G. One might be tempted to ask whether each such dominating path is, again, already an edge-dominating path of G, but a very simple example shows that this assumption can not be true. Both graphs given in Figure 2 are asteroidal triple free, but as one can check easily, the first one has a dominating path that is not edge-dominating, and the second does not have any edge-dominating path at all. Both of the examples do contain edge-dominating trails though, and, as we will see in the following, this is true for all asteroidal triple-free graphs. First, we have to give two technical lemmata.

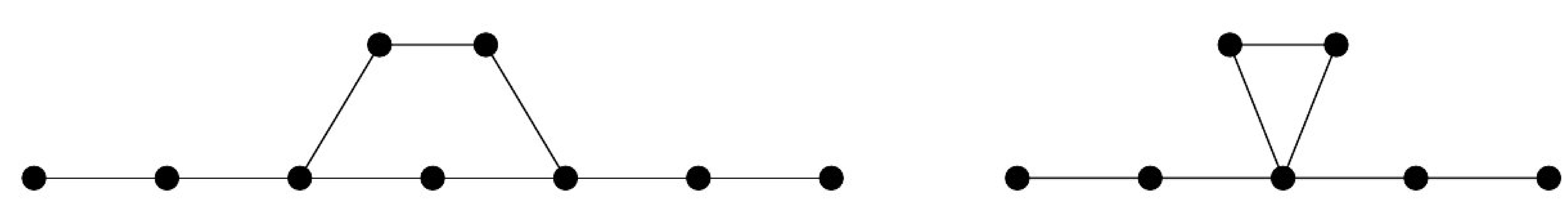

Figure 2: AT-free graphs, with dominating path that is not edge-dominating.

Lemma 3.2 *Let $G = (V, E)$ be an AT-free graph, (r, s) a dominating pair of G, T an r, s-trail of G and $K = x_0, x_1, \ldots, x_{j-1}, x_j = x_0$, $j > 0$, a chordless cycle of G with $x_i \notin T$ for all $i \in \{0, \ldots, j\}$. Then we can construct an r, s-trail T' such that*

(i) $\mathrm{length}(T') \geq \mathrm{length}(T) + \mathrm{length}(K)$,

(ii) the vertex set of T' contains only vertices of T and K,

(iii) x_0 and x_1 are contained in T'.

Proof. (In the following indices of vertices in K are always considered to be modulo the size of K.) T is a dominating trail, therefore for each $x_i \in K$ there is some $z \in T$, that dominates x_i. Since G is AT-free, K can have size 3, 4, or 5.

(1) Suppose there is some $z \in T$ that is adjacent to some $x_i, x_{i+1} \in K$. Then we construct T' by replacing z in T by $z, x_{i+1}, x_{i+2}, \dots, x_{i-1}, x_i, z$. Hence we can assume that there is no such z in T.

(2) Suppose there are vertices z, z' in T with $(z, z') \in E$ and z dominates x_i, z' dominates x_{i+1}. If $(z, z') \in T$, we construct T' by replacing the subtrail z, z' of T by $z, x_i, x_{i-1}, \dots, x_{i+2}, x_{i+1}, z'$. If, on the other hand, $(z, z') \notin T$ we define T' by replacing z by $z, x_i, x_{i-1}, \dots, x_{i+2}, x_{i+1}, z', z$. Hence, we can assume that there is no such pair of vertices z, z' in T.

(3) Suppose there is some $z \in T$ that is adjacent to some $x_{i-1}, x_{i+1} \in K$. If $i \notin \{0, 1\}$ then we construct T' by replacing z by $z, x_{i+1}, x_{i+2}, \dots, x_{i-2}, x_{i-1}, z$. Thus we can assume that there is no such z for $i \notin \{0, 1\}$. Suppose there is a $z \in T$ for $i = 0$, i.e. that is adjacent to x_{-1} and x_1. Because of (1), the size of K is greater than 3 and if K has size 4, z is adjacent to x_{i-1}, x_{i+1} for $i = 2$, which was ruled out earlier. Hence the size of K is 5. By (2), vertices z, z_2, z_3 form an independent set of G. Since the only neighbors of z in K are x_1 and x_4, the only neighbor of z_3 in K is x_3 and z_2 is not adjacent to x_4, vertices z, z_2, z_3 form an AT of G, which is a contradiction. Analogously one can show that we can assume that there is no z for $i = 1$. Hence we can assume that there is no $z \in T$ that is adjacent to some $x_{i-1}, x_{i+1} \in K$.

Consequently we can restrict ourselves to the following three cases:

Case 1. Size of K is 3. By (1) and (2) vertices z_0, z_1, z_2 form an independent triple and, together with K, they form an AT of G. Hence this case can not occur.

Case 2. Size of K is 4. Let $z_0, \dots, z_3$ be vertices of T that dominate $x_0, \dots, x_3$ correspondingly. By (1) and (3) each z_i dominates only x_i and by (2) $(z_i, z_{i+1}) \notin E$. If there is a pair z_i, z_{i+2} with $(z_i, z_{i+2}) \notin E$, vertices z_i, z_{i+1}, z_{i+2} form an AT of G. Therefore $(z_0, z_2), (z_1, z_3) \in E$. If both edges are not contained in T, then we can construct T' by replacing z_0 by $z_0, x_0, x_1, z_1, z_3, x_3, x_2, z_2, z_0$. If one of the edges, say (z_0, z_2) is contained in T but $(z_1, z_3) \notin T$ then we can construct T' by replacing the subtrail z_0, z_2 of T by $z_0, x_0, x_1, z_1, z_3, x_3, x_2, z_2$. If both edges are contained in T and, without loss of generality, $T = X, z_0, z_2, Y, z_1, z_3, Z$ (where X, Y, Z are subtrails of T), then we can define T' by $T' = X, z_0, x_0, x_1, z_1, Y^{-1}, z_2, x_2, x_3, z_3, Z$.

Case 3. Size of K is 5. Let $z_0, \dots, z_4$ be vertices of T that dominate $x_0, \dots, x_4$ correspondingly. By (1) and (3) each z_i dominates only x_i and by (2) $(z_2, z_3) \notin E$. Consequently x_0, z_2, z_3 form an AT. Hence this case can not occur.

It is easy to check that for the above constructed T' (i), (ii), and (iii) are satisfied. □

Lemma 3.3 *Let $G = (V, E)$ be an AT-free graph, (r, s) a dominating pair of*

G, T an r,s-trail of G, $C = x_0, x_1, \ldots, x_{k-1}, x_k = x_0$, $k > 0$, a circuit of G and T, C disjoint. Then we can construct an r,s-trail T'' such that

$$\text{length}(T'') \geq \text{length}(T) + \text{length}(C).$$

PROOF. Let $R = x_i, x_{i+1}, \ldots, x_{i+j}$ be a subpath of C that is chordless in G but with $(x_i, x_{i+j}) \in E$. We define the chordless cycle K of G by $K = x_i$, x_{i+1}, …, x_{i+j}, x_i and consider three cases.

Case 1. $x_{i+j+1} = x_i$. In this case K is a subtrail of C, and the removal of K from C leaves a circuit $C' = x_0, x_1, \ldots, x_i, x_{i+j+2}, \ldots, x_k$. By Lemma 3.2 (i), we can construct an r,s-trail T' with $\text{length}(T') \geq \text{length}(T) + \text{length}(K)$. By the fact that C has no vertex in common with T and by Lemma 3.2 (ii), it follows that none of the edges of C' is contained in T'. Therefore we can construct T'' by replacing x_i by $x_i, x_{i+1}, \ldots, x_{i-1}, x_i$. Consequently

$$\text{length}(T'') \geq \text{length}(T) + \text{length}(K) + \text{length}(C') = \text{length}(T) + \text{length}(C).$$

Case 2. $x_{i+j+1} \neq x_i$ but $(x_i, x_{i+j}) \in C$. In this case there is some $h \in \{0, \ldots, k\}$ with $h \neq i$, $h \neq i+j$ and either $x_h = x_i$, $x_{h+1} = x_{i+j}$ or $x_h = x_{i+j}$, $x_{h+1} = x_i$. Without loss of generality $h > i+j$. Using Lemma 3.2 we construct an r,s-trail T' with $\text{length}(T') \geq \text{length}(T) + \text{length}(K)$.

If $x_h = x_i$, $x_{h+1} = x_{i+j}$ we can define a circuit C' that contains exactly those edges of C that are not contained in K: $C' = x_0$, x_1, …, x_i, x_{h-1}, …, x_{i+j}, x_{h+2}, …, x_k By Lemma 3.2 (iii), we can define T'' by inserting C' appropriately into T'. Similarly as in Case 1, none of the edges of C' is contained in T'. Hence

$$\text{length}(T'') \geq \text{length}(T) + \text{length}(C).$$

If, on the other hand, $x_h = x_{i+j}$, $x_{h+1} = x_i$ we can define two edge-disjoint circuits C'_1 and C'_2 that contain exactly those edges of C, that are not contained in K: $C'_1 = x_0, x_1, \ldots, x_i, x_{h+2}, \ldots, x_k$, $C'_2 = x_{i+j}, x_{i+j+1}, \ldots, x_h$. By Lemma 3.2 (iii), x_i and x_{i+j} are contained in T'. To define T'' we replace x_i in T' by $x_i, x_{h+2}, \ldots, x_k, x_0, \ldots, x_i$ and x_{i+j} by $x_{i+j}, x_{i+j+1}, \ldots, x_{h-1}, x_{i+j}$. Thus

$$\begin{aligned}\text{length}(T'') &\geq \text{length}(T) + \text{length}(K) + \text{length}(C'_1) + \text{length}(C'_2)\\ &= \text{length}(T) + \text{length}(C).\end{aligned}$$

Case 3. $x_{i+j+1} \neq x_i$ and $(x_i, x_{i+j}) \notin C$. Using Lemma 3.2 we construct an r,s-trail T' with $\text{length}(T') \geq \text{length}(T) + \text{length}(K)$. If $(x_i, x_{i+j}) \in T'$ we can define T'' by replacing (x_i, x_{i+j}) in T' by $x_i, x_{i-1}, \ldots, x_{i+j+1}, x_{i+j}$. Hence

$$\begin{aligned}\text{length}(T'') &= \text{length}(T') - 1 + \text{length}(x_i, x_{i-1}, \ldots, x_{i+j+1}, x_{i+j})\\ &\geq \text{length}(T) + \text{length}(K) - 1 + \text{length}(x_i, x_{i-1}, \ldots, x_{i+j+1}, x_{i+j})\\ &= \text{length}(T) + \text{length}(C).\end{aligned}$$

If $(x_i, x_{i+j}) \notin T'$ we can define T'' by replacing x_i by $x_i, x_{i-1}, \ldots, x_{i+j+1}, x_{i+j}, x_i$. (By Lemma 3.2 (iii) x_i is contained in T'.) Thus

$$\begin{aligned}\text{length}(T'') &= \text{length}(T') + \text{length}(x_i, x_{i-1}, \ldots, x_{i+j+1}, x_{i+j}) + 1\\ &\geq \text{length}(T) + \text{length}(K) + \text{length}(x_i, x_{i-1}, \ldots, x_{i+j+1}, x_{i+j}) + 1\\ &\geq \text{length}(T) + \text{length}(C).\end{aligned}$$

This completes the proof. □

Now we can state the main theorem of this section.

Theorem 3.4 *Every connected AT-free graph has an edge-dominating trail.*

PROOF. Let G be an AT-free graph and (r, s) a dominating pair of G, i.e. every path between r and s is a dominating path. Let T be an arbitrary r, s-path. Since (r, s) is a dominating pair, T is a dominating path and also a dominating trail.

In the following we describe a procedure which, given an r, s-trail T and an edge $e = (x, y)$, that is not edge-dominated by T, produces an r, s-trail T' that edge-dominates e and has a greater size than T. Consequently one can apply this procedure until an edge-dominating r, s-trail of G is created.

Now let T be an arbitrary r, s-trail and (x, y) an edge of G that is not edge-dominated by T. If x and y have a common neighbor a on T, we know that neither (x, a) nor (y, a) are contained in T and we create T' by replacing a in T by a, x, y, a. Hence we can assume x and y not to have a common neighbor on T.

Let $S = a, x_0, x_1, \ldots, x_k, b$ be a subsequence of T with shortest length, such that $(x, a), (y, b) \in E$ but $(x, x_i), (y, x_i) \notin E$ for all $i \in 0, \ldots, k$. Because S is the shortest sequence with the above properties, neither a nor b is an inner vertex of S. If $k = 0$ we can define T' by replacing S by a, x, y, b. Hence we can assume k to be greater than 0 and, of course, $(a, b) \notin T$. If $(a, b) \notin T$ but $(a, b) \in E$ we can define T' be replacing S by a, b, y, x, S. Hence we can assume $(a, b) \notin E$.

Now suppose that $x_0 \neq x_k$. If $(x_0, b) \in T$ then we can define T' by replacing S by a, x, y, b and (x_0, b) by $x_0, x_1, \ldots, x_k, b$. If $(x_0, b) \notin T$ but $(x_0, b) \in E$ we can replace S by $a, x, y, b, x_0, x_1, \ldots, x_k, b$. Hence we can assume that $(x_0, b) \notin E$. Analogously we can assume $(a, x_k) \notin E$. But this implies that vertices a, y, x_k form an AT of G. Hence from now on we can assume $x_0 = x_k$.

Let C be the circuit defined by $x_0, x_1, \ldots, x_{k-1}, x_k$. Obviously $k \geq 3$. If any of the vertices x_i ($i \in \{0, \ldots, k\}$) occurs on T outside of S we can define T' by replacing S by a, x, y, b and x_i by $x_i, x_{i+1}, \ldots, x_{i-i}, x_i$. Hence we can assume that none of the x_i occurs on T outside of S.

Now we define an r, s-trail T'' by replacing S by a, x, y, b. Then

$$\text{length}(T'') + \text{length}(C) > \text{length}(T). \tag{1}$$

T'' is an r, s-trail and C and T'' are disjoint. Hence we can apply Lemma 3.3 and get an r, s-trail T' with $\text{length}(T') \geq \text{length}(T'') + \text{length}(C)$. Consequently

$$\text{length}(T') \geq \text{length}(T'') + \text{length}(C) > \text{length}(T).$$

This completes the proof. □

As a direct consequence of the above proof, we can give an algorithm that, given an asteroidal triple-free graph G, computes an edge-dominating trail of it.

Theorem 3.5 *There is a $O(nm)$ algorithm to compute an edge-dominating trail in an AT-free graph.*

PROOF. The proofs of Theorem 3.4, Lemma 3.2 and Lemma 3.3 give a procedure for creating an edge-dominating trail in an AT-free graph. At first one selects a dominating pair (r, s) and an arbitrary r, s-path T which is the initial r, s-trail. This can be done in linear time (see [6]). After that, for each non-dominated edge (x, y) a new r, s-trail T' is constructed. Each time a new trail T' is constructed this new trail contains two vertices (x and y) that were not contained in T and a closer look at the above proofs reveals that at most one of the vertices of T is not contained in T'. Hence in each step of the construction the new trail contains at least one more vertex than the previous trail. Consequently, at most n construction steps are required.

In each single step of the construction first a non-dominated edge has to be found. For this the algorithm has to step through T and, for each vertex that was previously not found on this T, must mark all incident edges as dominated. To identify an non-dominated edge one simply scans the list of all edges until one finds an non-dominated edge. Clearly this takes linear time. To identify the shortest subsequence $S = a, x_0, x_1, \ldots, x_k, b$ with $(a, x) \in E, (b, y) \in E$ but $(x, x_i), (y, x_i) \notin E$, for $i = 0 \ldots, k$ (see proof of Theorem 3.4), we mark all neighbors of x with an X and all neighbors of y with a Y and than simply scan through T, storing the shortest subsequence of T starting with an X-vertex and ending with a Y-vertex or vice versa. Again, this takes linear time. All remaining exchange steps, described in the proof can, as well, be implemented in linear time. Hence the complexity of this algorithm is $O(nm)$. $\square$

Again, we can apply Lemma 2.5, and obtain the following corollary.

Corollary 3.6 *Let G be an AT-free graph.* L(G) *is traceable if and only if G is connected.*

In order to construct hamiltonian cycles in line graphs of asteroidal triple-free graphs it would be nice to have a similar result on the existence of edge-dominating circuits, as for edge-dominating trails. For the 2-connected case, this is quite simple (see Lemma 3.7). For the general case a little more work has to be done (see Theorem 3.8).

Lemma 3.7 *Let G be a 2-connected graph. If G is AT-free it contains an edge-dominating circuit.*

PROOF. Let (r, s) be a dominating pair of G. By MENGER's Theorem, there are two openly disjoint r, s-paths P_1, P_2 in G. If we remove the inner vertices of P_2 from G the remaining graph G' is still connected (because P_1 is a dominating path of G) and (r, s) is a dominating pair of G'. Hence, by Theorem 3.4 there is an edge-dominating r, s-trail T in G' and by connecting T to P_2 we obtain an edge-dominating circuit of G. $\square$

Theorem 3.8 *Let G be a connected AT-free graph. G has an edge-dominating circuit if and only if it has no non-trivial bridge.*

PROOF. It is easy to see that, if G does contain a non-trivial bridge, it cannot have an edge-dominating circuit.

Now let G be an AT-free graph that does not contain a non-trivial bridge and let (r, s) be a dominating pair of G. If r or s has degree 1 select the corresponding neighbor instead. Let $A = \{x_1, x_2, \ldots, x_k\}$ be the set of all articulation points of G. Every r, s-path is dominating G, therefore every such path has to contain all vertices of A. Consequently every vertex of A is an r, s-separator and ordering of the vertices of A on any r, s-path is always the same. Because the vertices of A are the only articulation points of G and because there is no non-trivial bridge in G, it is easy to see that between any two consecutive vertices x_i, x_{i+1} of A there are two openly disjoint paths $P_1(x_i, x_{i+1})$, $P_2(x_i, x_{i+1})$. Similarly there are openly disjoint paths $P_1(a, x_1)$, $P_2(a, x_1)$ and $P_1(x_k, b)$, $P_2(x_k, x_b)$ (if any of the paths has no inner vertex make it the P_1 path). If we remove the inner vertices of each of the P_2 paths the remaining graph G' is still connected and (r, s) is a dominating pair of G'. By Theorem 3.4 G' has an edge-dominating r, s-trail P. Adding to P the path formed by all the P_2 paths of G creates an edge-dominating circuit of G. $\square$

Corollary 3.9 *Let G be a AT-free graph.* $\mathrm{L}(G)$ *is hamiltonian if and only if G is connected and does not contain a non-trivial bridge.*

Applying Theorem 3.5, Theorem 3.8 and Corollary 3.6, Corollary 3.9 gives the following algorithmic result.

Theorem 3.10 *Given a connected AT-free graph $G = (V, E)$ with $n = |V|$, $m = |E|$, there is an $O(n + m)$ algorithm that checks whether its line graph* $\mathrm{L}(G)$ *contains a hamiltonian path or a hamiltonian cycle, and there is an $O(nm)$ algorithm to compute a hamiltonian path and a hamiltonian cycle for* $\mathrm{L}(G)$*, if it exists.*

4 Hamiltonicity in claw-free AT-free graphs

The RYJÁČEK-*closure* $\mathrm{cl}(G)$ of a given graph G is the graph that arises by subsequent completion of the neighborhood of some vertex v of G if this neighborhood induces a connected graph. Consequently, for each vertex of $\mathrm{cl}(G)$ the neighborhood induces either a complete graph or a disconnected graph.

For the so defined closure concept, RYJÁČEK [13], and RYJÁČEK et al. [2] could show the following properties.

Lemma 4.1 ([2, 13]) *Let G be a claw-free graph, then*

(i) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph,*

(ii) *the length of the longest path of G is equal to the longest path of* $\mathrm{cl}(G)$*, and*

(iii) *the length of the longest cycle of G is equal to the length of the longest cycle of* $\mathrm{cl}(G)$*.*

In a series of publications (see e.g. [3, 4]) different subclasses of claw-free graphs were examined for their *stability* properties with respect to the RYJÁČEK-closure. As shown in the following, both asteroidal triple-free graphs and, their generalizations, graphs of bounded asteroidal number, are stable under the RYJÁČEK-closure.

Lemma 4.2 *Let G be an AT-free graph. If G' arises from G by completing the neighborhood of some vertex c of G then G' is AT-free.*

PROOF. Suppose G' does contain an AT x, y, z. For one of the vertices of the AT, say y, there is a chordless path P in G' between the remaining two vertices, x, z, that avoids the neighborhood of y, whereas in G all paths between x and z use vertices of $\mathrm{N}(y)$. P contains at least one new edge, say (a, b), that was put into G' by the completion of $\mathrm{N}(c)$ and, since P is chordless, P contains not more than one new edge. Now we distinguish two cases. Either vertex c is a neighbor of y. That implies $(y, a), (y, b) \in E(G')$, which is a contradiction to P avoiding the neighborhood of y. Or c is not in the neighborhood of y. In this case there is a path $\tilde{P}$ in G that avoids the neighborhood of y, again a contradiction. □

Theorem 4.3 *The class of AT-free graphs is stable under the* RYJÁČEK*-closure.*

PROOF. By Lemma 4.2 in each step of the construction of the RYJÁČEK-closure of a given AT-free graph, the property to be AT-free is preserved. Consequently $\mathrm{cl}(G)$ is AT-free. □

For a given graph G, an independent set of vertices S is called *asteroidal set* if for each $x \in S$ the set $S - \{x\}$ is in one connected component of the graph $G - (\mathrm{N}(x) \cup \{x\})$. The asteroidal number of a graph G is defined as the maximum cardinality of an asteroidal set of G, and is denoted by $\mathrm{an}(G)$ [11].

Lemma 4.4 *Let G be a graph with* $\mathrm{an}(G) \leq k$*. If G' arises from G by completing the neighborhood of some vertex c of G then* $\mathrm{an}(G') \leq k$*.*

PROOF. Suppose G' does contain an asteroidal set A of size k. For at least one of the vertices of A, say y, there are chordless paths in G' between any pair of vertices of $A - \{y\}$, that avoids the neighborhood of y, whereas in G there are two non-empty sets of vertices A_1, A_2, with $A - \{y\} = A_1 \cup A_2$ such that the vertices of A_1 and A_2 are in different connected components of $G - \mathrm{N}[y]$.

Let x be a vertex of A_1 and z a vertex of A_2 and let P be a chordless x, z-path in G' that avoids the neighborhood of y. P contains at least one new edge (a, b) that was put into G' by the completion of the neighborhood of some vertex c and, since P is chordless, there is no other new edge in P. If c is a neighbor of y, both a and b are neighbors of y in G', which is a contradiction to P avoiding the neighborhood of y. Hence c is not in the neighborhood of y. Consequently, there is a path P' in G, that avoids the neighborhood of y. This is a contradiction to x, z being in different connected components of $G - \mathrm{N}[y]$. This shows that $\mathrm{an}(G') \leq k$. $\square$

Theorem 4.5 *The class of graphs G with* $\mathrm{an}(G) \leq k$ *is stable under the* RYJÁČEK-*closure.*

PROOF. By Lemma 4.4 in each step of the construction of the RYJÁČEK-closure of a given graph G, the asteroidal number of G is not increased. Consequently $\mathrm{an}(\mathrm{cl}(G)) \leq \mathrm{an}(G)$. $\square$

From Theorem 4.3 we can draw the following corollary.

Corollary 4.6 *Let G be a claw-free AT-free graph.*

(i) G is traceable if and only if G is connected and

(ii) G is hamiltonian if and only if G is 2-connected.

PROOF. It is easy to see that every traceable graph is connected and every hamiltonian graph is 2-connected.

Suppose G is a connected/2-connected claw-free AT-free graph and $\mathrm{cl}(G)$ its RYJÁČEK-closure. In each step of the closure process, only edges are added, hence $\mathrm{cl}(G)$ remains connected/2-connected and by Lemma 4.3 AT-free. By Lemma 4.1 (i) $\mathrm{cl}(G)$ is a line graph. Applying Theorem 2.6 shows that $\mathrm{cl}(G)$ is traceable/hamiltonian, and, by Lemma 4.1 (ii)/(iii) G is hamiltonian. $\square$

The results of Corollary 4.6 were previously shown by Duffuss et al. [9] and Shepherd [14] for a larger class of graphs, the CN-free graphs. CN-free graphs are graphs that neither containing a claw nor a net (the six vertex graph consisting of a K_3 and a single pendant vertex on each of the vertices of the K_3) as an induced subgraph.

In [12] it has been proved that every 3-connected, not necessarily claw-free, and AT-free graph G on at least $3k + 2$ vertices contains a connected subgraph H on $k + 1$ vertices such that $G - V(H)$ is 2-connected. From this it follows, using Corollary 4.6, that every 3-connected claw-free AT-free graph G has cycles of any length greater or equal to $(2|G| - 1)/3$ — so it is $(2|G| - 1)/3$-pancyclic. (By a Theorem of SHEPHERD [14], such graphs are indeed 3-pancyclic).

A very simple corollary of the above results is the following.

Corollary 4.7 *Let G be a claw-free AT-free graph. The length of the longest cycle of G is equal to the order of the largest block of G.*

5 AT-free graphs of small maximum degree

In this section we shall apply Theorem 3.8 to AT-free graphs with small maximum degree.

Theorem 5.1 *Suppose that G is a 2-connected AT-free graph with no vertex of degree exceeding* 3. *Then G is one of the graphs in Figure 3.*

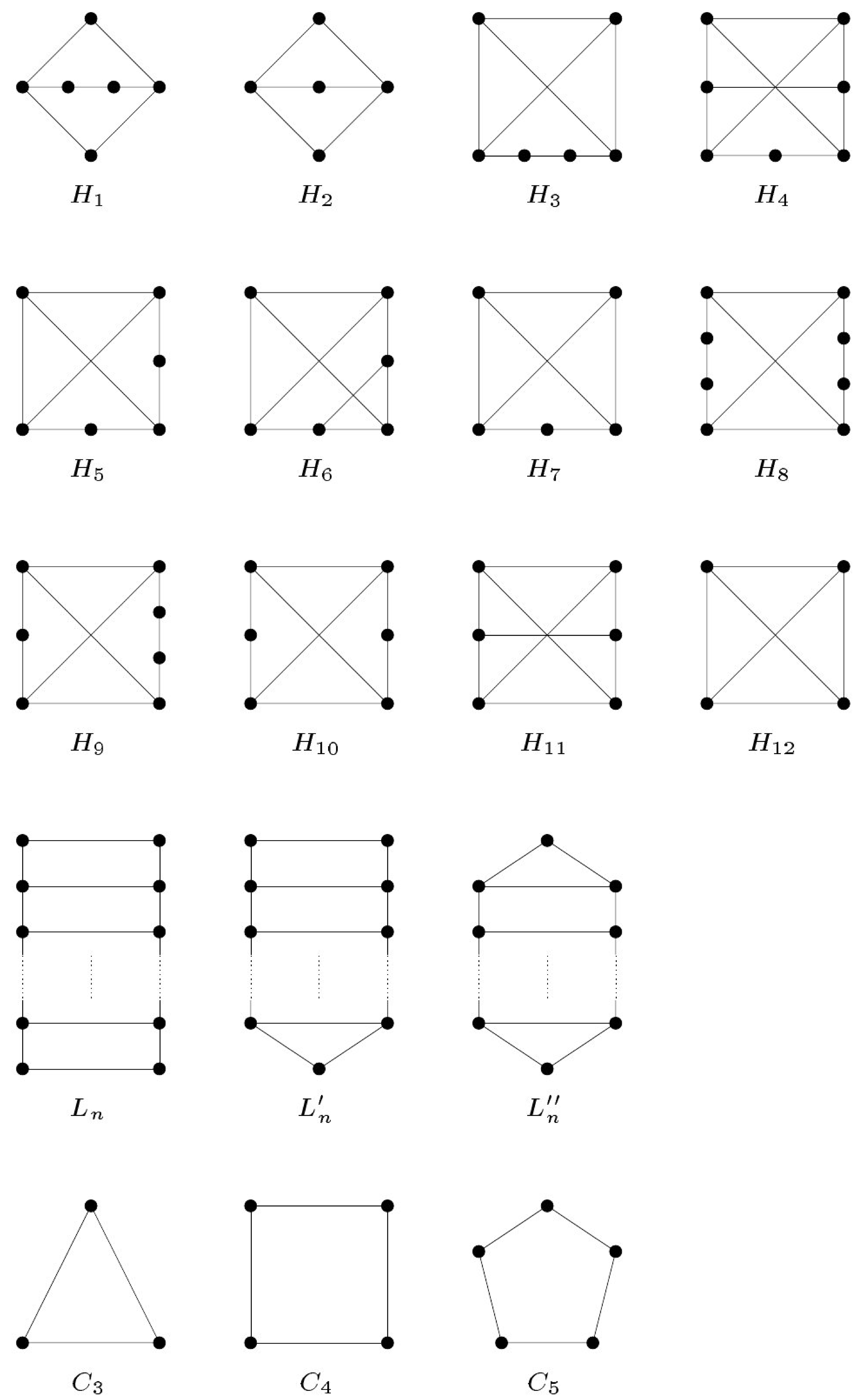

Figure 3: 2-connected AT-free graphs with no vertex of degree exceeding 3.

Proof. First note that all of the graphs mentioned above are non-isomorphic and AT-free. By Theorem 3.8, G has an edge-dominating circuit. Among all edge-dominating circuits we choose one with maximum number of vertices.

Case 1. Suppose that there exists a vertex x of degree 2 in $V(G) - V(C)$. Let $x_1, \ldots, x_\ell,\ y_1, \ldots, y_m,\ \ell \leq m$ be the two openly disjoint subpaths of C between the neighbors $x_1 = y_1, x_\ell = y_m$ of x in $V(C)$. We have $\ell, m \leq 4$, for otherwise x_1, x_3, x_ℓ or y_1, y_3, y_m would form an AT. By choice of C, we have $\ell, m \geq 3$.

Case 1a. If $\ell = m = 4$ then there must be an edge between $\{x_2, x_3\}$ and $\{y_2, y_3\}$. Therefore, if there would not be an edge between x_2, y_3 then x, x_2, y_3 form an AT. By symmetry, there is also the edge (x_3, y_2). But then $V(G) = V(C) \cup \{x\}$, and $V(G)$ is hamiltonian, contradicting the choice of C.

Case 1b. If $\ell = 3$ and $m = 4$ then $V(G) = V(C) \cup \{x\}$ (for otherwise there would be a vertex y incident to at least two of x_2, x_3, y_2. By choice of C, y is not adjacent to both x_2, x_3, and, consequently, adjacent to y_2 and one of x_2, x_3. In particular, $V(G) = V(C) \cup \{x, y\}$, and again G is hamiltonian, contradicting the choice of C). Since G is non-hamiltonian, it must be the graph H_1 obtained from $K_{2,3}$ by subdividing a single edge once.

Case 1c. If $\ell = m = 3$ then again $V(G) = V(C) \cup \{x\}$, similarly as in Case 1b. It follows that G is the graph $H_2 = K_{2,3}$.

Case 2. Suppose that there exists a vertex x of degree 3 in $V(G) - V(C)$. By choice of C, the neighbors of x in $V(C)$ are independent, so they form an AT.

Case 3. The first two cases do not occur. Then C is a hamiltonian cycle. For each chord e, we may choose two induced cycles C_e, C'_e in the graph $C + e$ such that $E(C_e) \cap E(C'_e) = \{e\}$. We say that distinct chords e, f *cross*, if $E(C_e) \cap E(C_f) \neq \emptyset$ and $E(C_e) \cap E(C'_f) \neq \emptyset$. (This does not depend on the choice of C_e, C'_e.)

Case 3a. There exists a pair of crossing chords e, f. Let $x_{i,1}, \ldots, x_{i,n(i)}$, $i \in \{1, 2, 3, 4\}$, be a partition of C into four openly disjoint subpaths such that $x_{1,1} = x_{4,n(4)}, x_{3,1} = x_{2,n(2)}$ are the end vertices of e and $x_{2,1} = x_{1,n(1)}, x_{4,1} = x_{3,n(3)}$ are the end vertices of f. For $i \in \{1, 2, 3, 4\}$, we have $n(i) \leq 4$, for otherwise $x_{1,i}, x_{3,i}, x_{n(i),i}$ would form an AT. Furthermore, $n(i) \geq 3$ does not hold for *all* $i \in \{1, 2, 3, 4\}$, for otherwise $x_{1,1}, x_{2,1}, x_{3,2}$ would form an AT. Thus, without loss of generality, $n(1) = 2$.

Case 3a1. If $n(3) = 4$ and $n(2) \geq 3$ or $n(4) \geq 3$ then $x_{2,1}, x_{3,1}, x_{3,3}$ or $x_{1,1}, x_{4,1}, x_{3,n(3)-2}$ would form an AT.

Case 3a2. If $n(3) = 4$ and $n(2) = n(4) = 2$ then G is then graph H_3 obtained from a K_4 by subdividing a single edge twice.

Case 3a3. If $n(3) = 3$ and $n(2) = 4$ or $n(4) = 4$ then, by symmetry, we may assume that $n(4) = 4$. It follows that $x_{3,1}, x_{4,1}, x_{4,3}$ form an AT.

Case 3a4. If $n(3) = 3$ and $n(2) = n(4) = 3$ then the graph induced by $x_{2,2}, x_{3,2}, x_{4,2}$ must contain (exactly) one edge, for otherwise these three vertices would form an AT. If there is an edge between $x_{2,2}$ and $x_{2,4}$ then G must be the graph H_4 obtained from a C_6 with three pairwise crossing chords by subdividing a non-chordal edge, and thus, by symmetry, we may assume that $x_{3,2}$ and $x_{4,2}$ are adjacent. But then $x_{2,1}, x_{3,1}, x_{4,2}$ form an AT.

Case 3a5. If $n(3) = 3$ and $n(2) = 3 \wedge n(4) = 2$ or $n(2) = 2 \wedge n(4) = 3$ then G must be the graph H_5 obtained from a K_4 by subdividing two incident edges once, or the graph H_6 obtained from H_5 by adding an edge between the vertices of degree 2 in H_5. Both H_5, H_6 are AT-free.

Case 3a6. If $n(3) = 3$ and $n(2) = n(4) = 2$ then G must be the graph H_7 obtained from a K_4 by subdividing one edge once, which is AT-free.

Case 3a7. If $n(3) = 2$ and $n(2) = n(4) = 4$ then there are no chords between $\{x_{2,2}, x_{2,3}\}$ and $\{x_{4,2}, x_{4,3}\}$, for otherwise there would be an AT in G. So G must be the graph H_8 obtained from a K_4 by subdividing two independent edges twice each.

Case 3a8. If $n(3) = 2$ and $n(2) = 3$, $n(4) = 4$, or $n(2) = 4$, $n(4) = 3$, then we G must be the graph H_9 obtained from a K_4 by subdividing two independent edges once, twice, respectively, or the graph H_4 obtained from H_9 by adding a further edge between two vertices of degree 2

Case 3a9. If $n(3) = 2$ and $n(2) = n(3) = 3$ then G is the graph H_{10} obtained from a K_4 by subdividing two independent edges once, or G is the graph H_{11} obtained from H_{10} by adding a single edge. Both H_{10}, H_{11} are AT-free.

Case 3a10. If $n(3) = 2$ and $n(2) = 2 \wedge n(4) \geq 3$ or $n(2) \geq 3 \wedge n(4) = 2$ then G must be the graph H_3 or H_7.

Case 3a11. If $n(3) = n(2) = n(4) = 2$ then G is the graph H_{12}, a K_4.

Case 3b. G has a chord but no pair of crossing chords. We show that in this case G must be a *ladder* $L_n = P_n \times K_2$ for some $n \geq 3$, or the graph L'_n obtained from L_n by contracting exactly one edge with both end vertices having degree 2, or the graph L''_n obtained from L_n by contradicting both edges which have end vertices of degree 2. First note that G contains a vertex of degree 2, say $x_1 = y_1$. Without loss of generality, we may assume that $x_0 \in C_e$ for all chords e of C. Since C has no crossing chords, there exists an enumeration $e_1, \ldots, e_n$ of the chords of C such that $V(C_{e_1}) \subseteq V(C_{e_2}) \subseteq \ldots \subseteq V(C_{e_n})$. We have $V(C_{e_{i+1}}) - V(C_{e_i}) = V(e_{i+1})$, for otherwise the two neighbors of $V(e_i)$ in $V(C) - V(C_{e_i})$ were not adjacent and thus would form an AT in G together with x_0. For the same reason, applied to $i = n$, $G - V(C_{e_n})$ consists of one or two adjacent vertices of degree 2 in G, so $|V(C'_{e_n})| \leq 4$. By applying the same arguments to C'_{e_i} with reverse orderings and starting with some vertex x'_0 of $V(C'_{e_n})$, we obtain $|V(C_{e_1})| \leq 4$. Consequently, G of the form L_n, L'_n, or L''_n for some $n \geq 3$.

Case 3c. G has no chords. Then G must be a graph C_3, C_4, C_5. □

Most of the exceptional graphs have more than two vertices of degree 2; these can not occur as blocks in a 3-regular AT-free graphs. This observation enables us to characterize the AT-free 3-regular graphs immediately:

Theorem 5.2 *We call a graph of the preceding theorem k-*valent *if it has exactly k vertices of degree 2 and the graph obtained from it by linking k further, new vertices to the vertices of degree 2 by a matching is still AT-free. So the 0-valent graphs are H_6, H_{11}, and H_{12}, the 1-valent graphs are H_4, H_7, and the 2-valent graphs are H_5, H_{10}, and L''_n, $n \geq 3$.*

Let G be a connected 3-regular AT-free graph. Then G is either a 0-valent graph or can be obtained from vertex disjoint copies of two 1-valent graphs G_0, G_{n+1} and n 2-valent graphs $G_1, \ldots, G_n$, $n \geq 0$ by adding the edges (y_i, x_{i+1}) for $i \in \{0, n\}$, where $V_2(G_0) = \{y_0\}$, $V_2(G_{n+1}) = \{x_n\}$, and $V_2(G_i) = \{x_i, y_i\}$ for $i \in \{1, \ldots, n\}$. (Note that x_i, y_i are contained in the same orbit of G_i.)

References

[1] J. Bondy and U. Murty, *Graph Theory with Applications*, MacMillan, 1976.

[2] S. Brandt, O. Favaron, and Z. Ryjáček, *Closure and stable hamiltonian properties in claw–free graphs*, Tech. Rep. No. 97, University of West Bohemia, November 1996.

[3] J. Brousek, Z. Ryjáček, and O. Favaron, *Forbidden subgraphs, hamiltonicity and closure in claw–free graphs.* submitted, May 1998.

[4] J. Brousek, Z. Ryjáček, and I. Schiermeyer, *Forbidden subgraphs, stability and hamiltonicity.* submitted, February 1998.

[5] D. G. Corneil, S. Olariu, and L. Stewart, *Asteroidal triple–free graphs*, Tech. Rep. 262/92, Department of Computer Science, University of Toronto, June 1992.

[6] ———, *Computing a dominating pair in an asteroidal triple-free graph in linear time*, in WADS: 4th Workshop on Algorithms and Data Structures, S.G.Akl, F. Dehne, J.-R.Sack, and N.Santoro, eds., Lecture Notes in Computer Science 955, Springer-Verlag, 1995, pp. 358–368.

[7] ———, *Asteroidal triple–free graphs*, SIAM Journal of Discrete Mathematics, 10 (1997), pp. 399–430.

[8] R. Diestel, *Graph Theory*, Graduate Texts in Mathematics 173, Springer, 1997.

[9] D. Duffuss, M. Jacobson, and R. Gould, *Forbidden subgraphs and the hamiltonian theme*, in Proceedings, 4th Int. Conference on the Theory and Applications of Graphs, 1980, pp. 297–316.

[10] F. Harary and C. Nash-Williams, *On eulerian and hamiltonian graphs and line graphs*, Canad. Math. Bull., 8 (1965), pp. 701–710.

[11] T. Kloks, D. Kratsch, and H. Müller, *Asteroidal sets in graphs*, in Graph–Theoretic Concepts in Computer Science, Proceedings of WG'97, R. H. Möhring, ed., LNCS 1335, Springer-Verlag, 1997, pp. 229–241.

[12] M. Kriesell, *Contractible subgraphs in 3-connected graphs*, Preprint 570, Technische Universität Berlin, 1997.

[13] Z. RYJÁČEK, *On a closure concept in claw–free graphs*, Journal of Combinatorial Theory, Series B, 70 (1997), pp. 217–224.

[14] F. SHEPHERD, *Hamiltonicity in claw–free graphs*, Journal of Combinatorial Theory, Series B, 53 (1991), pp. 173–194.

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